The stability of a two-dimensional laminar jet

By T. TATSUMI
*Cavendish Laboratory, University of Cambridge**

and T. KAKUTANI
Department of Physics, University of Kyoto

(*Received* 21 *January* 1958)

SUMMARY

This paper deals with the stability of a two-dimensional laminar jet against the infinitesimal antisymmetric disturbance. The curve of the neutral stability in the (α, R)-plane (α, the wave-number; R, Reynolds number) is calculated using two different methods for the different parts of the curve; the solution is developed in powers of $(\alpha R)^{-1}$ for obtaining the upper branch of the curve and in powers of αR for the lower branch.

The asymptotic behaviour of these branches is that for branch I, $\alpha \to 2$, $c \to \frac{2}{3}$ for $R \to \infty$; and for branch II, $R \sim 1{\cdot}12\alpha^{-1/2}$, $c \sim 1{\cdot}20\alpha^2$ for $\alpha \to 0$. Some discussion is given on the validity of the basic assumption of the stability theory in relation to the numerical result obtained here.

1. INTRODUCTION

A large part of the theory of hydrodynamical stability, as it has been developed in the past, is concerned with the question whether an *essentially parallel flow* is stable or not against *infinitesimal disturbances*. Within this limitation the theory has been successful in predicting the stability characteristics of a number of typical laminar flows, its results having been supported satisfactorily by experiments.

The application of this theory, however, has hitherto been confined to flows with at least one solid boundary, and little is known about the stability of boundary-free flows such as a laminar jet, wake, and mixing region between parallel flows. This may not be surprising in view of the fact that the free flow, having at least one inflexion point in its velocity profile, is supposed to be highly unstable so that its critical Reynolds number is very small. For such a small Reynolds number the usual methods of asymptotic approximation for large Reynolds number become ineffective and a new method of approximation must be introduced. The more fundamental difficulty of the boundary-free case, however, lies in that the basic assumption of the hydrodynamical stability theory that the undisturbed flow is approximately parallel is not satisfied for smaller Reynolds numbers, because then we can

* Now on leave from University of Kyoto, Japan.

neglect neither the lateral velocity component nor the change in the longitudinal velocity profile. Thus, the investigation of the stability of free flows under the framework of the existing stability theory appears to lead to self-contradictory results. The present calculation on the two-dimensional laminar jet seems to confirm this anticipation; the critical Reynolds number is found to be only 4·0, associated with the wave-number $\alpha = 0{\cdot}2$.

The stability of the laminar jet has been already dealt with by Curle (1957), using a new method of approximation which was devised by McKoen (1957) and himself, especially for investigating the stability of free flows. Their method is essentially to reduce the fourth-order disturbance equation to a second-order one, using some physical considerations. However, to neglect the fourth-order term in the equation does not seem allowable for small wave-number even when the Reynolds number is fairly small. Moreover, Curle's way of approximating the unknown stream function by the linear combination of two inviscid solutions is quite arbitrary and a different choice of the combination factor may lead to a different value of the critical Reynolds number.

The principle of the calculation adopted in this paper is quite simple. Solutions are expanded into inverse power series of αR for large value of αR, and in ascending series of αR for small αR. In practice, solutions are calculated up to the second- and third-order term for the above series respectively. Unfortunately, owing to the slow convergency of the series, we have not been able to work out the entire curve of neutral stability in the (α, R)-plane. But the branch of the curve for $\alpha \to 0$ is sufficient for determining the critical Reynolds number.

2. Formulation of the problem

The basic flow under consideration is the two-dimensional laminar jet ejected from an infinite line orifice. The velocity distribution of this flow was calculated by Schlichting (1933) and Bickley (1937), giving the result

$$\left.\begin{aligned} u &= U_0 \operatorname{sech}^2\left(\frac{y}{L}\right), \\ v &= (\tfrac{3}{4}R_x)^{-2/3} U_0 \left\{\frac{2y}{L}\operatorname{sech}^2\left(\frac{y}{L}\right) - \tanh\left(\frac{y}{L}\right)\right\}, \end{aligned}\right\} \tag{2.1}$$

where

$$U_0 = \left(\frac{3M^2}{32\nu x}\right)^{1/3}, \qquad L = \left(\frac{M}{48\nu^2 x^2}\right)^{-1/3}, \tag{2.2}$$

and

$$R_x = \left(\frac{Mx}{\nu^2}\right)^{1/2}, \tag{2.3}$$

$$M = \int_{-\infty}^{\infty} u^2\, dy = \text{const.} \tag{2.4}$$

represent the Reynolds number and the kinetic momentum flux of the flow, respectively.

It may be seen easily from (2.1) that for large Reynolds number R_x the ratio v/u $(= O(R_x^{-2/3}))$ is small, so that the flow becomes nearly parallel. Moreover, since from (2.1) $(\partial u/\partial x)/u = O(x^{-1})$, the variation of u with x is also small for large values of x which are associated in general with large R_x. Thus, the fundamental premise of the hydrodynamical stability theory that

$$\partial u/\partial x = 0, \qquad v = 0, \tag{2.5}$$

is approximately satisfied in this flow for large R_x. Now let us assume that the condition (2.5) is satisfied, expecting R_x in the present calculation to be fairly large. It will be found later that the final result of the calculation does not always confirm this assumption, but this will be discussed in § 7.

We take U_0 and L defined by (2.2) as the representative velocity and length of the flow, respectively, and make all variables non-dimensional with respect to these characteristic quantities. Denoting the dimensionless coordinates again by (x, y), the non-dimensional velocity profile is given from (2.1) by

$$U(y) = \operatorname{sech}^2 y. \tag{2.6}$$

It is convenient to define another Reynolds number by

$$R = U_0 L/\nu \tag{2.7}$$

which is related with R_x by

$$R = (\tfrac{9}{2})^{1/3} R_x^{2/3}. \tag{2.8}$$

The relation (2.8) shows that the discussion of the previous paragraph on the condition (2.5) is equally valid when R_x there is replaced by R.

By virtue of Squire's theorem (Squire 1933) we need consider only the two-dimensional disturbance, whose stream function may be defined by

$$\psi = \phi(y) e^{i\alpha(x-ct)}, \tag{2.9}$$

where α, being real and positive, represents the wave-number of the disturbance, and c is in general a complex constant. The real part c_r of c represents the phase velocity of the disturbance whereas αc_i, c_i being the imaginary part, is the amplification factor. According as the values of c_i are positive, zero, and negative, the basic flow becomes unstable, neutrally stable, and stable, respectively.

Substituting (2.9) into the Navier–Stokes equations and neglecting the non-linear terms with respect to ϕ, we obtain the Orr–Sommerfeld equation for the disturbance:

$$(U-c)(\phi'' - \alpha^2\phi) - U''\phi = \frac{1}{i\alpha R}(\phi^{\text{iv}} - 2\alpha^2\phi'' + \alpha^4\phi), \tag{2.10}$$

where the primes represent differentiation with respect to y.

The boundary condition for the disturbance is given by the requirement that all disturbance velocities must vanish at infinity, that is,

$$\phi' = \alpha\phi = 0 \qquad \text{at } y = \pm\infty. \tag{2.11}$$

When $U(y)$ is an even function of y, as in this case, we can consider the antisymmetric and symmetric disturbances separately. Since the antisymmetric disturbance is, on physical grounds, supposed to be more unstable than the symmetric one, and this is the case for flows with solid boundaries, we shall deal here only with the former disturbance. For antisymmetric disturbances, the boundary condition (2.11) reduces to

$$\phi'(\infty) = \alpha\phi(\infty) = 0, \qquad \phi'(0) = \phi'''(0) = 0, \tag{2.12}$$

and we need consider only the semi-infinite range $y \geqslant 0$.

3. Solutions for large αR

The analytical property of the solutions of the Orr–Sommerfeld equation (2.10) for large values of αR has been investigated in detail by previous researchers (see Lin 1955, ch. 8). Its four particular solutions are classified into two *inviscid solutions* and two *viscous solutions*. While the former two have non-vanishing values throughout the flow field for the limit $\alpha R \to \infty$, the latter two vanish almost everywhere except in the immediate vicinity of the solid walls. We need not therefore consider these viscous solutions in the present problem, where there is no solid boundary, so long as we are dealing with large values of αR. Foote & Lin (1950) clarified this point mathematically and concluded that the effect of viscosity enters the stability problem of the unlimited flow only through the higher approximations of the inviscid solutions, and that the boundary condition at the origin $\phi'(0) = \phi'''(0) = 0$ (for the even solution) reduces to $\phi'(0) = 0$. We therefore deal here only with the inviscid solutions which are obtained in the form

$$\phi(y) = \sum_{n=0}^{\infty} (\alpha R)^{-n} \phi^{(n)}(y; \alpha, c). \tag{3.1}$$

Substituting (3.1) into (2.10) and equating terms of the same powers of αR on both sides, we obtain the following equations:

$$\left.\begin{aligned}(U-c)(\phi^{(0)''} - \alpha^2\phi^{(0)}) - U''\phi^{(0)} &= 0,\\ (U-c)(\phi^{(n)''} - \alpha^2\phi^{(n)}) - U''\phi^{(n)} &= -i(\phi^{(n-1)\mathrm{iv}} - 2\alpha^2\phi^{(n-1)''} + \alpha^4\phi^{(n-1)}),\end{aligned}\right\} \tag{3.2}$$

for $n \geqslant 1$.

The solution of (3.2) is first calculated in three separate regions of the flow field: (i) the region far outside of the core of jet, (ii) the region near the critical layer at which $y = y_c$, and (iii) the region around the centre. Then, on joining these solutions analytically at two boundary points of these three regions, we obtain the solution over the whole flow field.

(i) *Outer solution.* For large y the velocity profile may be written in the form

$$U(y) = 4e^{-2y} \sum_{m=0}^{\infty} (-1)^m (m+1) e^{-2my}, \tag{3.3}$$

so that

$$\frac{U''}{U-c} = \sum_{m=1}^{\infty} A_m e^{-2my}, \tag{3.4}$$

where

$$A_1 = -\frac{16}{c}, \qquad A_2 = \frac{4}{c}(1-2c)A_1,$$

$$A_3 = \frac{1}{c^2}(16-40c+27c^2)A_1, \quad A_4 = \frac{4}{c^3}(16-48c+46c^2-16c^3)A_1, \quad \ldots .$$

Substituting (3.4) into (3.2) and taking account of the first two boundary conditions of (2.12), we obtain the following solutions for large y:

$$\left.\begin{aligned} \phi^{(0)} &\sim \Phi^{(0)} = C_1 e^{-\alpha y} \sum_{m=0}^{\infty} a_m e^{-2my}, \\ \phi^{(1)} &\sim \Phi^{(1)} = C_1 e^{-\alpha y} \sum_{m=1}^{\infty} b_m e^{-2my}, \end{aligned}\right\} \tag{3.5}$$

where C_1 is an integration constant and

$$a_0 = 1, \qquad a_1 = \frac{A_1}{4(1+\alpha)}, \qquad a_2 = \frac{A_1 a_1 + A_2}{8(2+\alpha)},$$

$$a_3 = \frac{A_1 a_2 + A_2 a_1 + A_3}{12(3+\alpha)}, \qquad \ldots;$$

$$b_1 = \frac{4i}{c}(1+\alpha)a_1,$$

$$b_2 = \frac{1}{8(2+\alpha)}\left[A_1 b_1 + \frac{64i}{c}\left\{(2+\alpha)^2 a_2 + \frac{(1+\alpha)^2}{c} a_1\right\}\right],$$

$$b_3 = \frac{1}{12(3+\alpha)}\left[A_1 b_2 + A_2 b_1 + \frac{16i}{c}\left\{9(3+\alpha)^2 a_3 + \frac{16}{c}(2+\alpha)^2 a_2 + \right.\right.$$
$$\left.\left. + \frac{8}{c^2}(2-c)(1+\alpha)^2 a_1\right\}\right], \quad \ldots .$$

(ii) *Critical-layer solution.* In the neighbourhood of the critical layer $y = y_c$, where $U(y_c) = c$, the solution is expanded in a power series of $(y-y_c)$. Substituting the Taylor series expansion

$$U - c = U_c'(y-y_c) + \frac{1}{2!}U_c''(y-y_c)^2 + \ldots$$

into the first equation of (3.2), we obtain a pair of independent particular solutions for $\phi^{(0)}$ as follows:

$$\left.\begin{aligned} \phi_1^{(0)} &\sim \Psi_1^{(0)} = C_2 \sum_{m=1}^{\infty} f_m (y-y_c)^m, \\ \phi_2^{(0)} &\sim \Psi_2^{(0)} = C_3 \left\{\sum_{m=0}^{\infty} g_m (y-y_c)^m + \frac{U_c''}{U_c'} \Psi_1^{(0)} \log(y-y_c)\right\}, \end{aligned}\right\} \tag{3.6}$$

where C_2 and C_3 are integration constants and

$$f_1 = 1, \qquad f_2 = \frac{U_c''}{2U_c'}, \qquad f_3 = \frac{\alpha^2}{6} + \frac{U_c'''}{6U_c'}, \qquad f_4 = \frac{\alpha^2}{9} f_2 + \frac{U_c^{\text{iv}}}{24U_c'}, \quad \ldots;$$

$$g_0 = 1, \qquad g_1 = 0, \qquad g_2 = \frac{\alpha^2}{2} + \frac{U_c'''}{2U_c'} - \left(\frac{U_c''}{U_c'}\right)^2,$$

$$g_3 = -\frac{\alpha^2}{18}\frac{U_c''}{U_c'} - \frac{U_c'' U_c'''}{6U_c'^2} + \frac{1}{12}\frac{U_c^{\text{iv}}}{U_c'} - \frac{1}{8}\left(\frac{U_c''}{U_c'}\right)^3, \quad \ldots .$$

The way of choosing the proper branch of $\log(y-y_c)$ in $\Psi_2^{(0)}$ has been established by earlier workers (see Lin 1955, p. 130) with the result that if $U_c' < 0$, as in this case,

$$\log(y-y_c) = \log|y-y_c| \qquad \text{for } y-y_c > 0,$$
$$\log(y-y_c) = \log|y-y_c|+i\pi \quad \text{for } y-y_c < 0.$$

The second-order solution $\phi^{(1)}$ is obtained from the second equation of (3.2) by quadrature and two particular solutions are expressed in terms of the $\phi^{(0)}$'s as follows:

$$\phi_j^{(1)} \sim \Psi_j^{(1)} = \Psi_1^{(0)} \int^y M_j \Psi_2^{(0)}\, dy - \Psi_2^{(0)} \int^y M_j \Psi_1^{(0)}\, dy, \tag{3.7}$$

where

$$M_j = -i(U-c)^{-1}(\Psi_j^{(0)\text{iv}} - 2\alpha^2 \Psi_j^{(0)''} + \alpha^2 \Psi_j^{(0)}), \qquad j = 1, 2.$$

The determination of the proper logarithmic branch must be carried out in the same way as for $\Psi_2^{(0)}$. It should be noted here that the solution $\Psi_2^{(1)}$ is $O\{(y-y_c)^{-2}\}$ in the neighbourhood of the critical point, so that the solution $\Psi_2^{(1)}$ loses its meaning there. Since, however, we are not concerned with the eigenfunction itself but only with the eigenvalues of α, R and c, a good approximation at the joining points of the solutions (y_1 and y_2 in § 4) is sufficient for the present purpose. In fact this is attained in the present case, because the joining points are fairly distant from the critical point and the coefficients of the singular terms in $\Psi_2^{(1)}$ are small for small $(c-c_s)/c_s$ (c_s is the limiting value of c for $\alpha R \to \infty$), and therefore the singular terms are insignificant at the joining points.

(iii) *Inner solution.* For small value of y the velocity distribution may be written in the form

$$U-c = \sum_{m=0}^{\infty} F_m y^{2m}, \tag{3.8}$$

where

$$F_0 = 1-c, \quad F_m = (-1)^m 2^{2(m+1)}(2^{2(m+1)}-1)\frac{2m+1}{(2m+2)!}B_{m+1}, \quad m \geqslant 1,$$

and the B's are the Bernoulli numbers.

Substituting (3.8) into (3.2) and taking the third boundary condition of (2.12) into account, we obtain the solution in the neighbourhood of the origin as follows:

$$\left.\begin{aligned} \phi^{(0)} &\sim \Omega^{(0)} = C_4 \sum_{m=0}^{\infty} h_m y^{2m}, \\ \phi^{(1)} &\sim \Omega^{(1)} = C_4 \sum_{m=1}^{\infty} k_m y^{2m}, \end{aligned}\right\} \tag{3.9}$$

where

$$h_0 = 1, \qquad h_1 = \frac{1}{2F_0}(\alpha^2 F_0 + 2F_1), \qquad h_2 = \frac{1}{12F_0}\{(\alpha^2 F_1 + 12F_2) + \alpha^2 F_0 h_1\},$$

$$h_3 = \frac{1}{30F_0}\{(\alpha^2 F_2 + 30F_3) + (\alpha^2 F_1 + 10F_2)h_1 + (\alpha^2 F_0 - 10F_1)h_2\},$$

$$h_4 = \frac{1}{56F_0}\{(\alpha^2 F_3 + 56F_4) + (\alpha^2 F_2 + 28F_3)h_1 + \alpha^2 F_1 h_2 + (\alpha^2 F_0 - 28F_1)h_3\}, \quad \dots;$$

$$k_1 = -\frac{i}{2F_0}(24h_2 - 4\alpha^2 h_1 + \alpha^4),$$
$$k_2 = \frac{1}{12F_0}\{-i(360h_3 - 24\alpha^2 h_2 + \alpha^4 h_1) + \alpha^2 F_0 k_1\},$$
$$k_3 = \frac{1}{30F_0}\{-i(1680h_4 - 60\alpha^2 h_3 + \alpha^4 h_2) + (\alpha^2 F_1 + 10F_2)k_1 + (\alpha^2 F_0 - 10F_1)k_2\},$$
$$\dots .$$

4. Boundary value problem for large αR

We now proceed to join the outer, critical-layer, and inner solutions analytically at some appropriate points, y_1 and y_2, say, each located on the boundaries of the three regions. Noting that the system of equations (3.2) is of the second order, we can easily see, extending Foote & Lin's (1950) analysis, that the effect of the viscous solution does not enter the eigenvalue problem and that the analytical continuation of the solution is simply carried out by making the functional value and the first-order derivative of the solution continuous at y_1 and y_2 instead of doing so up to the third-order derivatives. If this process is carried out the solution thus obtained will satisfy all the boundary conditions of (2.12), because the outer and inner solutions derived in § 3 already satisfy the respective boundary conditions for $y \to \infty$ and $y = 0$.

First, joining the outer and critical-layer solutions at y_1 ($y_c < y_1 < \infty$), we have

$$\frac{\Phi'(y_1)}{\Phi(y_1)} = \frac{\Psi_1'(y_1) + (C_3/C_2)\Psi_2'(y_1)}{\Psi_1(y_1) + (C_3/C_2)\Psi_2(y_1)}. \tag{4.1}$$

The similar relationship between the critical-layer and inner solutions is obtained by the continuation at y_2 ($0 < y_2 < y_c$):

$$\frac{\Omega'(y_2)}{\Omega(y_2)} = \frac{\Psi_1'(y_2) + (C_3/C_2)\Psi_2'(y_2)}{\Psi_1(y_2) + (C_3/C_2)\Psi_2(y_2)}. \tag{4.2}$$

Eliminating C_3/C_2 from (4.1) and (4.2), we obtain the eigenvalue equation relating the parameters α, R and c:

$$E(\alpha, R, c) = 0. \tag{4.3}$$

We are particularly interested in finding the condition for the neutral disturbance. Putting $c_i = 0$ in (4.3) and solving the complex equation, we can express α and R as functions of c

$$\alpha = \alpha(c), \qquad R = R(c). \tag{4.4}$$

The limiting case of the problem for $\alpha R \to \infty$ was solved by Savic (1941). According to his results,

$$\alpha = 2, \qquad U''(y_c) = 0,$$

so that

$$c = U(y_c) = \tfrac{2}{3},$$

and the solution (for the antisymmetric disturbance) is given by

$$\phi = \text{const.} \times \operatorname{sech}^2 y.$$

For large αR the corresponding value of c may be expected to be not far from the limiting value $c_s = \frac{2}{3}$. Then, for a number of the prescribed values of c in this neighbourhood we obtain an equation connecting α and R, and we can find the roots of α and R by trial and error. During this process we expand the coefficients of the critical-layer solutions into powers of $c - c_s$ and neglect the terms of higher order than $(c-c_s)^3$. It is found that

c	α	R
$\frac{2}{3}$ (C_s)	2 (α_s)	∞
0·66	1·97	737
0·65	1·93	270
0·64	1·88	161
0·63	1·84	116
0·62	1·80	91·9
0·61	1·76	76·7
0·60	1·71	66·5

Table 1.

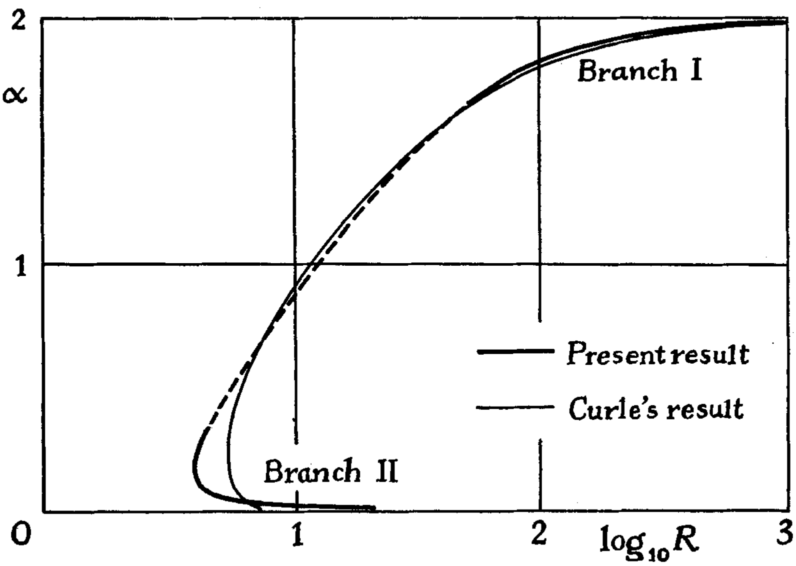

Figure 1. The curve of neutral stability in the (α, R)-plane.

the real solutions are obtainable only for those values of c smaller than c_s and that the approximation of the present solution becomes unsatisfactory for $c < 0{\cdot}6$. In the process of continuation of the solutions, the points $y_1 = y_c + 0{\cdot}2752$ and $y_2 = 0{\cdot}5503$ are used as the joining points, and the solutions are calculated up to the fourth term for $\Phi^{(0)}$, the fifth term for $\Omega^{(0)}$, and the seventh term for $\Psi_1'^{(0)}$ and $\Psi_2'^{(0)}$ respectively. The second-order solutions are calculated to the same order of approximation as the corresponding first-order solutions. The error caused by neglecting the remaining terms of each series is estimated to be at most 5 %. The numerical results are tabulated in table 1 and shown graphically in figure 1 (branch I) and figure 2.

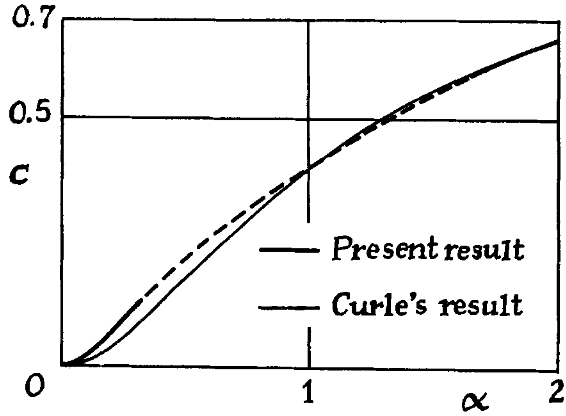

Figure 2. The curve of neutral stability in the (c, α)-plane.

5. Solutions for small αR

We now proceed to calculate the solutions for small values of αR. Equation (2.10) may be written as

$$(D^2-\alpha^2)(D^2-\alpha^2+i\alpha Rc)\phi = i\alpha R\{U(\phi''-\alpha^2\phi)-U''\phi\}, \qquad (5.1)$$

where $D = d/dy$. In the region far from the core of jet, that is, for large value of y, the velocity profile $U(y)$ almost vanishes as well as all its spatial derivatives. Thus, the right-hand side of (5.1) being negligible in this region, the four particular solutions of (5.1) reduce to

$$\phi = e^{\pm\alpha y}, \qquad e^{\pm\beta y}, \qquad \text{where } \beta^2 = \alpha^2 - i\alpha Rc. \qquad (5.2)$$

An approximate solution may be obtained by solving (5.1) with one of the solutions in (5.2) substituted on the right-hand side. Better solutions are derived by repeating this iteration process and we can express a formal solution for ϕ in the form

$$\phi(y) = \sum_{n=0}^{\infty} (i\alpha R)^n \phi^{(n)}(y;\alpha,\beta), \qquad (5.3)$$

where the ϕ's are related to each other by the following equations:

$$\left.\begin{aligned}&(D^2-\alpha^2)(D^2-\beta^2)\phi^{(0)} = 0,\\ &(D^2-\alpha^2)(D^2-\beta^2)\phi^{(n)} = U(\phi^{(n-1)''}-\alpha^2\phi^{(n-1)})-U''\phi^{(n-1)}, \quad n \geqslant 1.\end{aligned}\right\} \qquad (5.4)$$

The uniform convergence of the solution (5.3) for all αR can be shown as follows. Equation (5.1) together with the boundary conditions (2.12) is equivalent to the following integral equation:

$$\phi(y) = Ae^{-\alpha y}+\beta e^{-\beta y}+\frac{1}{c}\Big[-e^{-\alpha y}\int_{\infty}^{y} U'e^{\alpha y}\phi\, dy - e^{\alpha y}\int_{\infty}^{y} U'e^{-\alpha y}\phi\, dy +$$

$$+e^{-\beta y}\int_{\infty}^{y}\Big(U'+\frac{\beta^2-\alpha^2}{2\beta}U\Big)e^{\beta y}\phi\, dy +$$

$$+e^{\beta y}\int_{\infty}^{y}\Big(U'-\frac{\beta^2-\alpha^2}{2\beta}U\Big)e^{-\beta y}\phi\, dy\Big]$$

$$= Ae^{-\alpha y}+\beta e^{-\beta y}+$$

$$+i\alpha R\int_{y}^{\infty}\Big[U'(\eta)\frac{\sinh\{\frac{1}{2}(\alpha+\beta)(\eta-y)\}}{\frac{1}{2}(\alpha+\beta)}\frac{\sinh\{\frac{1}{2}(\alpha-\beta)(\eta-y)\}}{\frac{1}{2}(\alpha-\beta)}$$

$$+U(\eta)\frac{\sinh\{\beta(\eta-y)\}}{\beta}\Big]\phi(\eta)\, d\eta, \quad (5.5)$$

where A and B are integration constants. Changing the variable from y to $z = e^{-y}$ $(0 \leqslant z \leqslant 1)$, equation (5.5) takes the following form:

$$\phi(z) = f(z)+\lambda\int_{0}^{z} K(z,\zeta)\phi(\zeta)\, d\zeta, \quad (5.6)$$

where $\lambda = i\alpha R, \qquad f(z) = Az^{\alpha}+\beta z^{\beta},$

$$K(z,\zeta) = \Big[U'(\zeta)\frac{1}{\alpha^2-\beta^2}\Big\{\Big(\frac{\zeta}{z}\Big)^{\frac{1}{2}(\alpha+\beta)}-\Big(\frac{\zeta}{z}\Big)^{-\frac{1}{2}(\alpha+\beta)}\Big\}\times$$

$$\times\Big\{\Big(\frac{\zeta}{z}\Big)^{\frac{1}{2}(\alpha-\beta)}-\Big(\frac{\zeta}{z}\Big)^{-\frac{1}{2}(\alpha-\beta)}\Big\}-U(\zeta)\frac{1}{2\beta}\Big\{\Big(\frac{\zeta}{z}\Big)^{\beta}-\Big(\frac{\zeta}{z}\Big)^{-\beta}\Big\}\Big]\frac{1}{\zeta}.$$

The iteration process suggested in the previous paragraph leads to the solution in powers of λ

$$\phi(z) = f(z)+\lambda\int_{0}^{z} K(z,\zeta)f(\zeta)\, d\zeta+\sum_{m=2}^{\infty}\lambda^{m}\int_{0}^{z}K(z,\zeta_1)\, d\zeta_1\int_{0}^{\zeta_1}K(\zeta_1,\zeta_2)\, d\zeta_2\dots$$

$$\dots\int_{0}^{\zeta_{m-1}}K(\zeta_{m-1},\zeta_m)f(\zeta_m)\, d\zeta_m. \quad (5.7)$$

Now $\qquad |f(z)| \leqslant |A|+|B|,$

and since $U(\zeta) = 4\zeta^2(1+\zeta^2)^{-2}$,

$$|U(\zeta)/\zeta| < 4\zeta, \qquad |U'(\zeta)/\zeta| < 8\zeta.$$

Then, if $\alpha < 2$, $\mathscr{R}(\beta) < 2$,

$$|K(z,\zeta)| < N\Big(\frac{\zeta}{z}\Big)^{-p} \quad \text{for } 0 \leqslant \zeta \leqslant z,$$

where $p < 1$ and N is a finite constant. Hence

$$\Big|\int_{0}^{z} K(z,\zeta)f(\zeta)\, d\zeta\Big| < \frac{(|A|+|B|)N}{1-p},$$

so that

$$\Big|\int_{0}^{z} K(z,\zeta_1)\, d\zeta_1\dots\int_{0}^{\zeta_{m-1}}K(\zeta_{m-1},\zeta_m)f(\zeta_m)\, d\zeta_m\Big| < \frac{(|A|+|B|)}{1-p}\frac{(Nz)^m}{m!}$$

$$\leqslant \frac{(|A|+|B|)}{1-p}\frac{N^m}{m!}.$$

Thus, the series (5.7) as well as (5.3) converges uniformly for all values of λ ($= i\alpha R$), provided $\alpha < 2$ and $\mathscr{R}(\beta) < 2$.

The system of equations (5.4) can be solved successively by the method of variation of parameters, leading to the result as follows:

$$\left.\begin{aligned}
&\phi_1^{(0)} = e^{-\alpha y}, \quad \phi_2^{(0)} = e^{\alpha y}, \quad \phi_3^{(0)} = {}^{-\beta y}, \quad \phi_4^{(0)} = e^{\beta y};\\
&\phi_j^{(n)} = \frac{1}{i\alpha Rc}\bigg[e^{-\alpha y}\int^y Ue^{\alpha y}(\phi_j^{(n-1)\prime} + \alpha\phi_j^{(n-1)})\,dy + \\
&\quad + e^{\alpha y}\int^y Ue^{-\alpha y}(\phi_j^{(n-1)\prime} - \alpha\phi_j^{(n-1)})\,dy - \\
&\quad - e^{-\beta y}\int^y Ue^{\beta y}\left(\phi_j^{(n-1)\prime} + \frac{\alpha^2+\beta^2}{2\beta}\phi_j^{(n-1)}\right)dy - \\
&\quad - e^{\beta y}\int^y Ue^{-\beta y}\left(\phi_j^{(n-1)\prime} - \frac{\alpha^2+\beta^2}{2\beta}\phi_j^{(n-1)}\right)dy\bigg],
\end{aligned}\right\} \tag{5.8}$$

for $n \geqslant 1$, $j = 1, 2, 3, 4$. The general solution of (5.1) is given by

$$\phi = C_1\,\phi_1 + C_2\,\phi_2 + C_3\,\phi_3 + C_4\,\phi_4, \tag{5.9}$$

where the C's are arbitrary constants.

6. Boundary value problem for small αR

According to the first two conditions of (2.10), ϕ_2 and ϕ_4 are rejected and the lower limit of the integrals in (5.8) must be ∞. Then, in order that the second two conditions of (2.10) be satisfied by the solution (5.9) with not identically vanishing C_1 and C_3, ϕ_1 and ϕ_3 must satisfy the following condition:

$$\begin{vmatrix} \phi_1'(0) & \phi_3'(0) \\ \phi_1'''(0) & \phi_3'''(0) \end{vmatrix} = 0, \tag{6.1}$$

which gives the eigenvalue equation between the parameters α, R and c.

Substituting (5.8) into (6.1), the characteristic equation becomes

$$\begin{vmatrix} -\alpha^2+\beta^2+\sum\limits_{n=1}^{\infty}(i\alpha R)^n I^{(n)}(\phi_1) & \sum\limits_{n=1}^{\infty}(i\alpha R)^n I^{(n)}(\phi_3) \\ \sum\limits_{n=1}^{\infty}(i\alpha R)^n J^{(n)}(\phi_1) & \alpha^2-\beta^2+\sum\limits_{n=1}^{\infty}(i\alpha R)^n J^{(n)}(\phi_3) \end{vmatrix} = 0, \tag{6.2}$$

where

$$\left.\begin{aligned}
I^{(n)}(\phi_j) &= 2\int_0^\infty U\{\sinh(\alpha y)\phi_j^{(n-1)\prime} + \alpha\cosh(\alpha y)\phi_j^{(n-1)}\}\,dy,\\
J^{(n)}(\phi_j) &= 2\int_0^\infty U\left\{\sinh(\beta y)\phi_j^{(n-1)\prime} + \frac{\alpha^2+\beta^2}{2\beta}\cosh(\beta y)\phi_j^{(n-1)}\right\}dy,
\end{aligned}\right\} \tag{6.3}$$

We work out the solutions up to the second approximation and neglect terms of $i\alpha R$ of higher order than the third in equation (6.2). Then we have

$$\begin{aligned}
&-(\alpha^2-\beta^2)^2 + i\alpha R(\alpha^2-\beta^2)\{I^{(1)}(\phi_1) - J^{(1)}(\phi_3)\} + \\
&\quad + (i\alpha R)^2[(\alpha^2-\beta^2)\{I^{(2)}(\phi_1) - J^{(2)}(\phi_3)\} + \\
&\quad\quad + I^{(1)}(\phi_1)J^{(1)}(\phi_3) - I^{(1)}(\phi_3)J^{(1)}(\phi_1)] = 0,
\end{aligned} \tag{6.4}$$

where the I's and J's are calculated using (5.8) and (6.3).

In order to work out the integrations in (6.4), it is convenient to approximate the velocity profile $U(y)$ $(= 4e^{-2y}/(1+e^{-2y})^2)$ by a polynomial in e^{-2y}:

$$U(y) = \sum_{k=1}^{m} G_k e^{-2ky}.$$

In practice we adopt the following quartic

$$U(y) = 3{\cdot}87e^{-2y} - 6{\cdot}50e^{-4y} + 5{\cdot}61e^{-6y} - 1{\cdot}98e^{-8y}, \tag{6.5}$$

which fits the exact velocity profile quite satisfactorily as shown in table 2. Although the profile given by (6.5) does not satisfy the condition $U'(0) = 0$, this does not matter because all terms in (6.4) are expressed as integrals involving U alone. The complex equation (6.4) substituted from (6.5)

y	Exact $U(y)$	Approx. $U(y)$
0	1	1
0·10	0·9972	1·0086
0·22	0·9879	0·9973
0·36	0·9689	0·9748
0·51	0·9375	0·9372
0·69	0·8889	0·8875
0·92	0·8163	0·8163
1·20	0·7101	0·7115
1·61	0·5556	0·5557
2·31	0·3306	0·3274
$\int_0^\infty U\,dy$	1	0·9975
$\int_0^\infty dy \int_y^\infty U\,dy$	0·6931	0·6861

Table 2.

is rather cumbersome and therefore we again expand its second and third terms in powers of $\beta-\alpha$ $(=\alpha\sigma$, say) and neglect terms of higher order than $O(\sigma^3)$. Then, equation (6.4) takes the form

$$1 + iR\,V(\alpha,\sigma) - R^2\,W(\alpha,\sigma) = 0, \tag{6.6}$$

from which R is given by

$$R = V_r/W_i. \tag{6.7}$$

where the suffixes r and i refer to the real and imaginary parts of the quantity, respectively. Eliminating R between (6.6) and (6.7), we obtain the equation

$$1 - \frac{V_r}{W_i}V_i - \left(\frac{V_r}{W_i}\right)^2 W_r = 0, \tag{6.8}$$

from which σ (or β) is determined as a function of α. Substituting $\sigma(\alpha)$ into (6.7), $R(\alpha)$ is obtained, and finally $c(\alpha)$ is calculated from the relation

$$c = \frac{\alpha}{iR}\{1-(1+\sigma)^2\}. \tag{6.9}$$

A numerical calculation has been carried out for $\alpha = 0{\cdot}1$, $0{\cdot}2$, $0{\cdot}3$. For $\alpha \geqslant 0{\cdot}4$ the approximate equation (6.6) does not give a real root, and more laborious manipulation of (6.4) is necessary for obtaining the neutral curve for larger α. Numerical results are given in table 3 and figure 1 (branch II) and figure 2.

α	R	c
0·1	4·21	0·018
0·2	4·02	0·061
0·3	4·39	0·121

Table 3.

7. Discussion

It may be seen from figure 1 that for large Reynolds number R the wave-number α decreases monotonically with R, although the change in α is not remarkable for $R > 200$. The same tendency of the neutral curve was noticed by Lessen (1950) in the case of the mixing region. On the other hand, the upper branch of all existing neutral curves for the bounded flows behaves in a quite different manner; α increases with decreasing R. It may therefore be inferred that the viscosity acts on the unlimited flows as a decaying factor, whereas it has the dual effect of decaying and amplifying the disturbances of bounded flows, although the two examples only may not be sufficient for so concluding.

The lower branch of the neutral curve gives the critical Reynolds number

$$R = 4{\cdot}0 \quad \text{at } \alpha = 0{\cdot}2, \tag{7.1}$$

which, according to (2.8), corresponds to

$$R_x = 3{\cdot}77. \tag{7.2}$$

The numerical values of the critical Reynolds number given by (7.1) and (7.2) may not be very realistic, because for such small Reynolds numbers like these the basic assumption of this calculation (2.5) is not satisfied very well. For instance, the ratio of the lateral and longitudinal velocity components of the main flow becomes, from equation (2.1),

$$v/u = 0{\cdot}5 \times (\text{non-dimensional function})$$

for $R_x = 3{\cdot}77$, so that the second assumption of (2.5) can no longer be valid under such circumstances. On the other hand, the first assumption of (2.5) is not directly affected by the numerical value of the Reynolds number because the ratio $(\partial u/\partial x)/u$ depends only on x. Rigorously speaking, therefore, all that we can say is that (7.1) gives the critical Reynolds number of an artificial flow with the velocity profile (2.6) which must be maintained by applying some sort of body force. To treat the stability problem of the non-parallel flow in a more satisfactory manner seems to be beyond the scope of the existing theory of hydrodynamical stability. Having ourselves no new ideas at the moment for improving the existing theory so as to be able to deal with the more general case, we content ourselves with taking the values given by (7.1) and (7.2) as a qualitative estimate of the critical Reynolds number of the laminar jet flow.

The asymptotic behaviour of the lower branch can be obtained exactly without using the approximate formula (6.5). Retaining only the lowest order terms with respect to α in (6.4), we obtain

$$\beta(\alpha+\beta)+i\alpha R\{\alpha-\beta+3\beta(\alpha+\beta)\log 2\}+\alpha^2R^2\{\alpha-2\beta-3(\alpha-\beta)\log 2\} = 0, \tag{7.3}$$

where use is made of the relations $\int_0^\infty U\,dy = 1$, $\int_0^\infty Uy\,dy = \log 2$. If we solve this complex equation with the relation $\beta^2 = \alpha^2 - i\alpha Rc$, we obtain the asymptotic behaviour of the branch

$$\left.\begin{aligned} \alpha R^2 &= 1{\cdot}25 \quad \text{i.e.} \quad R = 1{\cdot}12\alpha^{-1/2}, \\ \text{and} \qquad c &= \{4(\alpha R^2)^{-1}-2\}\alpha^2 = 1{\cdot}20\alpha^2. \end{aligned}\right\} \tag{7.4}$$

With this relation (7.4) between α, R and c, it is confirmed that the higher order terms in $i\alpha R$ which are neglected in (6.4) do not produce any term of the same order as those in (7.3), and therefore (7.4) gives the exact asymptotic behaviour of the neutral curve for $\alpha \to 0$. The asymptotic branch (7.4) is shown graphically in figures 1 and 2.

The relation (7.4) shows that for $\alpha \to 0$, αR tends to zero whereas R itself increases indefinitely. This confirms the self-consistency of the expansion (5.3) of the solution in ascending powers of αR. It should be noted here that the so-called *inviscid solution*, which corresponds to the limit $\alpha R \to \infty$, has nothing to do with the lower branch of the present case which leads to the limiting 'inviscid' disturbance in the sense that $R \to \infty$, $\alpha R \to 0$ for $\alpha \to 0$. This asymptotic behaviour clearly disproves Curle's interpolation formula which expresses the unknown stream function as a linear combination of two inviscid solutions. It may also be interesting to note here that Pai (1951) found that the flow is unstable for the combination of large value of αR and small α. He concluded from this that there is no lower branch of the neutral curve or the lower branch is very close to the R-axis in the (α, R)-plane. According to the present result, only the latter of his conclusions can be correct.

In figures 1 and 2 the neutral curve calculated by Curle is shown for comparison. His curve agrees fairly well with the present result so far as the upper region of the (α, R)-plane is concerned. This is not very unexpected because his first assumption of neglecting ϕ^{iv} might be permissible for values of α which are not small and the possible error involved in his interpolation formula may also not be serious for small values of $(c-c_s)/c_s$. However, it may be too early to decide from this single example whether this agreement is entirely accidental or whether it gives some justification for McKoen and Curle's new technique, at least, for the neglect of the ϕ^{iv} term on the upper branch of the neutral curve.

The authors wish to express their cordial thanks to Professor S. Tomotika and Dr G. K. Batchelor for their guidance and encouragement throughout this work. One of the authors (T. T.) is also deeply grateful to Mr F. Ursell for his helpful suggestions, and the other (T. K.) to Professor K. Tamada, Mr H. Yosinobu and Mr K. Goto for their stimulating discussions. During the course of this work the authors have been in receipt of a grant-in-aid for fundamental scientific research from the Ministry of Education of Japan, and one of them (T. T.) has also been receiving the British Council Scholarship (1956–7).

References

Bickley, W. 1937 *Phil. Mag.* **23**, 727.

Curle, N. 1957 *Proc. Roy. Soc.* A, **238**, 489.

Foote, J. R. & Lin, C. C. 1950 *Quart. Appl. Math.* **8**, 265.

Lessen, M. 1950 *Nat. Adv. Comm. Aero., Wash., Rep.* no. 979.

Lin, C. C. 1955 *The Theory of Hydrodynamical Stability*. Cambridge University Press.

McKoen, C. H. 1957 *Aero. Res. Counc., Lond., Curr. Pap.* no. 303.

Pai, S. I. 1951 *J. Aero. Sci.* **18**, 731.

Savic, P. 1941 *Phil. Mag.* **32**, 245.

Schlichting, H. 1933 *Z. Angew. Math. Mech.* **13**, 260.

Squire, H. B. 1933 *Proc. Roy. Soc.* A, **142**, 621.